

Backwards 3 and 3 Forwards

or
Patterns your calculator
won't show you

Do we still teach children what division actually **is** - let alone how to do it, now that we have the pocket calculator ever-ready to work out our sums for us, or have we just lost the ability to do mental arithmetic since calculators were introduced? Some time ago a customer in front of me at the supermarket checkout wanted to pay for two tins of beer out of a four-pack. Neither the checkout operator nor the customer knew what to do to work out the price for the two tins, and a supervisor was called in to solve the problem. Should more lesson time be spent on fractions?

FORWARDS...

Division produces fractions; some schools push fractions to one side as being *old-fashioned* - decimals being considered *modern* -- but fractions can be more useful - especially in mental calculations - than decimals. If you were asked to work out $0.25 \times £400$, and a quarter of £400, wouldn't you find the fraction easier to use? And if you know that 25% is a quarter the same fraction will give you 25% of £400. And having said that, turning fractions into decimals produces interesting patterns; which is what I want to discuss here.

To turn $\frac{1}{2}$ into a decimal you divide the 1 by the 2 to get 0.5. To turn $\frac{1}{3}$ into a decimal, you divide the 1 by the 3 - and get 0.3333... where the figure 3 repeats without end. This 3 *recurs* because 3 is not a factor of ten. Divide 1 by 6 (for the fraction one-sixth), using pencil and paper (or calculator) and you get 0.16666... in which the six recurs. Obviously you can use a calculator for these divisions, but you won't always get what we might call a *complete* answer - try one-seventh on the average calculator and it will give you 0.1428571, while the pencil-and-paper calculation (divide until the pattern appears) produces 0.142857 142857 ... where the six-figure pattern (or *period*) repeats. Use a bigger calculator? or the computer? by all means, but they are limited by the number of figures they will print in an answer while the person working with pencil and paper is limited only by his own patience. One seventeenth, for example, has sixteen figures in its repeating period.

The *prime numbers* produce the basic patterns. In decimals the first few (ignoring 2 and 5, which both divide into ten), are 3, 7, 11, 13, 17.... and the figures in the periods of these are 1,6,2,6 and 16. If you look for a connection between the prime and its period you will find that the period is either one less than the prime (as 7 and 17), or a number which a factor of this. And, let us add, in base ten.

If you know anything about computers you will know that though they may print out answers in base ten (decimal), they work in base two (binary). Some calculators will convert whole numbers from base ten to base two, or four, or eight, or sixteen, but none will do division in bases other than ten. So, if we want to compare patterns for the primes in bases other than ten, we have to do it all ourselves - with pencil and paper. Any fraction can be written in the form $0.\dots$ in any number base, but we can only call them decimals if they're in base ten - so some people use the word "basimal" for all bases; those who use base twelve a lot refer to "dozenals" in that base.

As an example, in base five a seventh becomes $0.032412\ 032412\ \dots$ - again a six-figure recurring period, as in base ten, but you can't assume this will always be the case; in base four, for example, a seventh becomes $0.021\ 021$, with only three figures in the period. A fifth in base ten doesn't recur, because five is a factor of ten; but in base twelve the fifth will recur ($0.2497\ 2497\dots$) while the third and sixth will not.

ROUND AND ROUND...

Patterns don't just recur; they can also *circulate*. Going back to our fifth in base twelve, we have:

one fifth $0.2497\ 2497\ \dots$

two-fifths $0.4972\ 4972\ \dots$

three-fifths $0.7249\ 7249\ \dots$

four-fifths $0.9724\ 9724\ \dots$ - the figures repeat in the same order, but starting with a different one each time. This is called circulating, and happens when the period is a maximum (one less than the prime - 7 has 6 digits maximum, 5 has 4 and so on). If the period isn't of maximum length then more than one pattern emerges - and each often circulates - for example the sevenths in base four have two three-figure periods, 021 and 123, and the sevenths in order, start 021, 102, 123, 210, 231, 312 (i.e. six=sevenths in base four is $0.312\ 312\dots$ and the two periods are 021 and 123).

SYMBOLS...

You might like to work some of these out for yourself - in base ten and others. If you do, you will find one problem arises if you want to use a base greater than ten - apart from 0, there are only nine digits. For the computer world the letters A to F are used for the numbers ten to fifteen (when writing numbers in base sixteen); but before the computer made base sixteen important for storing binary numbers, text books used other characters - notably *t* for ten and *e* for eleven for use with base twelve (familiar in one form as shillings & pence, and feet & inches). Other symbols suggested (and in use) are ζ for ten and ϵ for eleven; these two were suggested by Sir Isaac Pitman. In base twelve, for example one-seventh is $0.186\zeta35\ 186\zeta35\dots$ and $1/11$ (a thirteenth...) is $0.0\epsilon\ 0\epsilon\ 0\epsilon\ \dots$

AND (FINALLY) BACKWARDS...

Did you ever use a "wind-up" calculator at school? The one with sliding pegs to set the numbers and a handle you could turn to add or subtract numbers? You

multiplied numbers by repeated addition - for example multiplying a number by 9 you would set the number and then turn the handle nine times forwards, or, and quicker, you could move the carriage on which the number was set one place left, turn the handle once forwards (giving **ten** times the number) and then move the carriage back to the units and turn the handle once backwards (taking **one** times the number off) to leave 9 times the number as your answer.

We can represent these actions with numbers - if we use negative digits. Some of you will have used logarithms and will remember the negative digits “bar one”, “bar two” and so on: $\bar{1}$, $\bar{2}$... We can write 9 as “ $1\bar{1}$ ” i.e. “ten less one unit” which represents the movement of the handle on the calculating machine. Apply these negative digits to our basimal patterns and we produce other interesting results. One-eleventh in base ten is $0.09\ 09\ 09\dots$ and becomes $0.1\bar{1}\ 1\bar{1}\ 1\bar{1}\ \dots$. In the first version the decimal gets closer and closer to the true value, approaching it “from below” while the version with negative digits alternately overshoots and comes back towards the true value. In other bases we find other patterns sharing this structure: $1/7$ in base twelve becomes $0.143\ \bar{1}\bar{4}\bar{3}\ 143\ \bar{1}\bar{4}\bar{3}$, and a fifth $0.25\ \bar{2}\bar{5}\ 25\ \bar{2}\bar{5}\dots$